33 pts

1 a) Inside well: $0 < x < a$

5 pts

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \Rightarrow \frac{d^2\psi}{dx^2} = -k^2\psi$$

$$k \equiv \frac{\sqrt{2mE}}{\hbar}$$

Solutions: $\boxed{\psi(x) = A\sin(kx) + B\cos(kx)}$ 1 pt

$\boxed{\psi(x=0) = 0 \Rightarrow B = 0}$ 1 pt

$\psi(x=a) = 0 \Rightarrow \sin(ka) = 0 \Rightarrow ka = n\pi \Rightarrow$ $\boxed{k = \frac{n\pi}{a}}$ 1 pt

$$\boxed{\psi_n(x) = A_n \sin\left(\frac{n\pi}{a}x\right)}$$ 1 pt

$$\frac{\sqrt{2mE}}{\hbar} = \frac{n\pi}{a} \Rightarrow \boxed{E = \frac{n^2\pi^2\hbar^2}{2ma^2}}$$ 1 pt

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7 pts

b) $|\psi\rangle = (3i|\psi_1\rangle + 2|\psi_2\rangle)A$ $\quad \langle H\rangle = ?$ $\quad \langle\psi|\psi\rangle = 1 \Rightarrow A = \frac{1}{\sqrt{13}}$

$$\langle H\rangle = \langle\psi|H|\psi\rangle = \frac{1}{13}\left(-3i\langle\psi_1| + 2\langle\psi_2|\right)H\left(3i|\psi_1\rangle + 2|\psi_2\rangle\right)$$

$$= \left(-3i\langle\psi_1| + 2\langle\psi_2|\right)\left(3iE_1|\psi_1\rangle + 2E_2|\psi_2\rangle\right)\frac{1}{13}$$

2 pts

2 pts $= \frac{(9E_1 + 4E_2)}{13} = \frac{9}{13}\frac{\pi^2\hbar^2}{2ma^2} + \frac{(4\cdot 4)}{13}\frac{\pi^2\hbar^2}{2ma^2}$

$$\langle H\rangle = \frac{25}{13}\frac{\pi^2\hbar^2}{2ma^2} \Rightarrow \boxed{\langle H\rangle = \frac{25}{26}\frac{\pi^2\hbar^2}{ma^2}}$$ 1 pt

Possible outcomes from a measurement of the energy are $E_1 = \frac{\pi^2\hbar^2}{2ma^2}$ and $E_2 = \frac{4\pi^2\hbar^2}{2ma^2}$. 2 pts

5 pts

c) $\Psi(x,t=0) = 3i\psi_1 + 2\psi_2$

$$\Psi(x,t) = 3i\, e^{-\frac{i\epsilon_1 t}{\hbar}}\psi_1 + 2\, e^{-\frac{i\epsilon_2 t}{\hbar}}\psi_2$$

-3 pts if not specify E1 & E2

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9 pts

d) $\langle H \rangle = \langle \psi_1 | H | \psi_1 \rangle = \boxed{\epsilon_1 \langle \psi_1 | \psi_1 \rangle}$ 2 pts $= \boxed{\epsilon_1}$ 2 pts

$\boxed{\sigma_H^2 = \langle H^2 \rangle - \langle H \rangle^2}$ 2 pts

$\langle H^2 \rangle = \langle \psi_1 | H \cdot H | \psi_1 \rangle = \epsilon_1^2 \langle \psi_1 | \psi_1 \rangle = \epsilon_1^2$

$\boxed{\sigma_H^2 = \epsilon_1^2 - \epsilon_1^2 = 0}$ 3 pts   As expected from a stationary state of the Hamiltonian

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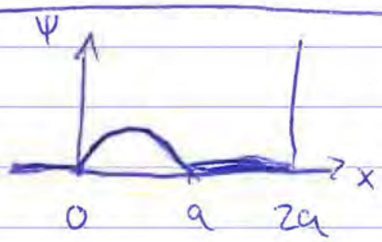
7 pts

e) $\psi(x) = A \sin\left(\frac{\pi x}{a}\right) \quad 0 < x < a$



New basis: $\boxed{\psi'_n = A'_n \sin\left(\frac{n\pi x}{2a}\right)}$ 2 pts

$\boxed{\psi = \sum_{n=1}^{\infty} c_n \psi'_n}$ 1 pt $\boxed{c_n = \int_{-\infty}^{\infty} \psi'^*_n \psi(x)\, dx}$ 1 pt

$\boxed{c_n = A'^*_n A \int_0^a \sin\left(\frac{n\pi x}{2a}\right) \sin\left(\frac{\pi x}{a}\right) dx =}$ 1 pt

$$= \left(\frac{1}{\frac{n^2\pi^2}{(2a)^2} - \frac{\pi^2}{a^2}}\right)\left[\left(\frac{\pi}{a}\right)\sin\left(\frac{n\pi}{2a}x\right)\cos\left(\frac{\pi x}{a}\right) - \left(\frac{n\pi}{2a}\right)\cos\left(\frac{n\pi}{2a}x\right)\sin\left(\frac{\pi x}{a}\right)\right]\Bigg|_{x=0}^{x=a}$$

$$= \left(\frac{1}{\left(\frac{\pi}{a}\right)^2\left(\frac{n^2}{2^2} - 1\right)}\right)\left[\left(\frac{\pi}{a}\right)\left(\sin\left(\frac{n\pi}{2}\right)\cos(\pi) - \cancelto{0}{\sin(0)}\cos(0)\right) - \left(\frac{n\pi}{2a}\right)\left(\cos\left(\frac{n\pi}{2}\right)\cancelto{0}{\sin(\pi)} - \cos(0)\cancelto{0}{\sin(0)}\right)\right]$$

$$C_n = \left(\frac{1}{\left(\frac{\pi}{a}\right)^2 \left(\frac{n^2}{\pi^2} - 1\right)} \right) \left(\frac{\pi}{a} \right) \sin\left(\frac{n\pi}{2} \right) (-1)$$

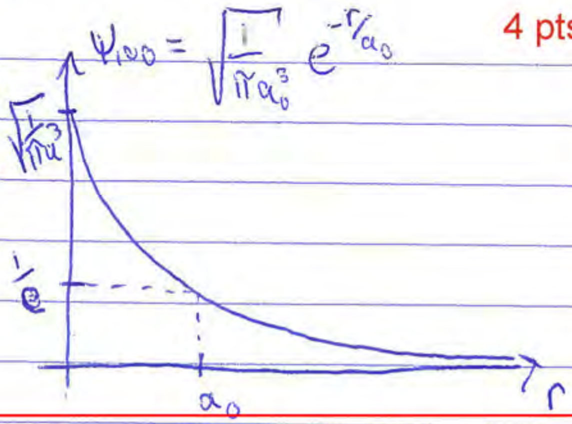
1 pt

$$= \begin{cases} 0 & \text{for } n = \text{even} \\ \text{non-zero} & \text{for } n = \text{odd} \end{cases}$$

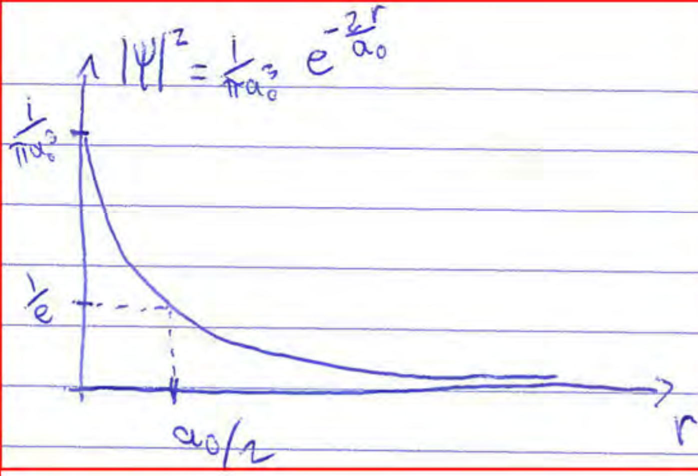
1 pt

33 pts

2 a) 6 pts



4 pts



Maximum value at $r=0$.
2 pts

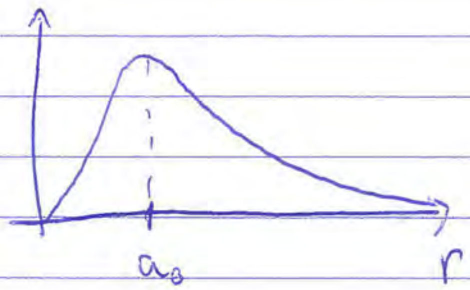
10 pts

b) Probability density: $\rho(r)d^3r = \boxed{|\Psi|^2 r^2 \sin\theta \, dr\, d\theta\, d\phi}$ 2 pts

$$\boxed{\rho d^3 r = \left(\frac{1}{\pi a_0^3}\right) e^{-2r/a_0} r^2 \sin\theta \, dr\, d\theta\, d\phi} \quad \text{3 pts}$$

Spherically symmetric, so take just the r component:

$$e^{-\frac{2r}{a_0}} r^2 = \begin{cases} 0 \text{ at } r=0 \\ 0 \text{ at } r \Rightarrow \infty \end{cases}$$



$$\boxed{\text{Maximum: } \frac{d}{dr}\left(e^{-\frac{2r}{a_0}} r^2\right) = 0} \quad \text{3 pts}$$

$$-\frac{2}{a_0} e^{-\frac{2r}{a_0}} r^2 + 2 r e^{-2r/a_0} = 0$$

2 pts

$$2r - \frac{2}{a_0} r^2 = 0 \Rightarrow 1 - \frac{1r}{a_0} = 0 \Rightarrow \boxed{r = a_0}$$

10 pts

c) $\langle r \rangle = ?$

$$\langle r \rangle = \iiint \Psi^* r \Psi \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$ 2 pt

$$= \left(\frac{1}{\pi a_0^3}\right)\left(\int_0^\infty e^{-\frac{2r}{a_0}} r^3 dr\right)\left(\int_0^\pi \sin\theta \, d\theta\right)\left(\int_0^{2\pi} d\phi\right)$$ 2 pt

$$= \left(\frac{1}{\pi a_0^3}\right)\left[\left(\frac{e^{-\frac{2r}{a_0}}}{\left(\frac{2}{a_0}\right)^4}\right)\left(\left(\frac{2}{a_0}\right)^3 r^3 + 3\left(\frac{2}{a_0}\right)^2 r^2 + 6\left(\frac{2}{a_0}\right) r + 6\right)\right]\Bigg|_{r=0}^{r=\infty}$$ 2 pts

$$\left(-\cos\theta \Big|_{\theta=0}^{\theta=\pi}\right)\left(2\pi\right)$$ 2 pts

$$= \left(\frac{4\pi}{\pi a_0^3}\right)\left[\emptyset - \left(\frac{1}{\left(\frac{2}{a_0}\right)^4}\right)\left(6\right)\right] = \frac{\cancel{4}}{\cancel{a_0^3}}\left(\frac{\cancel{6}^{3} \cancel{a_0^4}}{\cancel{16}_{\cancel{4}_{2}}}\right)$$

$$\boxed{\langle r \rangle = \frac{3}{2} a_0}$$ 2 pts

8 pts

d) Ground state for two electrons:

$$\boxed{\Psi = \underbrace{\psi^{a}_{100}\,\psi^{b}_{100}}_{\text{symmetric}}\;\underbrace{\chi_{\text{singlet}}}_{\text{antisymm.}}} \quad \text{1 pt}$$

1 pt

$$\boxed{\chi_{\text{singlet}} = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)}$$

total state is antisymmetric 2 pts

"bosonic electrons" → total state is symmetric 2 pts

$$\boxed{\Psi = \psi^{a}_{100}\,\psi^{b}_{100}\,\chi_{\text{triplet}}} \quad \text{1 pt}$$

1 pt

$$\boxed{\chi_{\text{triplet}} = \begin{cases} |\uparrow\uparrow\rangle \\ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ |\downarrow\downarrow\rangle \end{cases}}$$

34 pts

(3) $|x\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad |y\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

8 pts a) $\boxed{|s\rangle = \cos\phi\,|x\rangle + \sin\phi\,|y\rangle \quad \text{4 pts}}$

$\boxed{\text{2 pts} \quad \langle s|s\rangle = 1} = \cos^2\phi\,\langle x|x\rangle + \sin^2\phi\,\langle y|y\rangle$

$\boxed{\langle s|s\rangle = \cos^2\phi + \sin^2\phi = 1 \quad \text{2 pts}}$

---

7 pts

b) $|x\rangle$ photon passing through y-polarizer:

$\boxed{P = |\langle y|x\rangle|^2 = 0 \quad \text{3 pts}}$

2 pts

45° polarizer: $\boxed{P = \left|\frac{1}{\sqrt{2}}\left(\langle x| + \langle y|\right)|x\rangle\right|^2}$

$= \left|\frac{1}{\sqrt{2}}\left(\overset{1}{\cancel{\langle x|x\rangle}} + \overset{0}{\cancel{\langle y|x\rangle}}\right)\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 \Rightarrow P = \frac{1}{2}$ or 50%

2 pts

8 pts

c) $|x\rangle$ photon: first 45° pol then y-pol

$P_{45°} = \frac{1}{2}$ (calculated in b) 2 pts

Type text here

photon is now in state $|p\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle + |y\rangle\right)$ 2 pts

$$P_{y\text{-pol}} = \left| \langle y| \frac{1}{\sqrt{2}}\left(|x\rangle + |y\rangle\right)\right|^2 = \left|\frac{1}{\sqrt{2}}\left(\cancel{\langle y|x\rangle}^{\,0} + \cancel{\langle y|y\rangle}^{\,1}\right)\right|^2$$

2 pts $P_{y\text{-pol}} = \frac{1}{2}$ $\Rightarrow$ total prob. $P = \left(P_{45°}\right)\left(P_{y\text{-pol}}\right)$

$$P = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \Rightarrow \boxed{P = \frac{1}{4} \text{ or } 25\%}$$

2 pts

5 pts

d) $\frac{1}{\sqrt{2}}\left(|x\rangle|y\rangle + |y\rangle|x\rangle \right)$

2 pts

This is an entangled state. If you measure the polarization of one photon you know the state of the other. 3 pts

6 pts

e) Both photons passing through x-pol.

$$P = \left| {}_1\langle x| \, {}_2\langle x| \frac{1}{\sqrt{2}}\left(|x\rangle_1 |y\rangle_2 + |y\rangle_1 |x\rangle_2 \right) \right|^2$$ 2 pts

$$= \left| \frac{1}{\sqrt{2}}\left(\underbrace{{}_1\langle x|x\rangle_1}_{=1} \cdot \underbrace{{}_2\langle x|y\rangle_2}_{=0} + \underbrace{{}_1\langle x|y\rangle_1}_{=0} \cdot \underbrace{{}_2\langle x|x\rangle_2}_{=1} \right) \right|^2$$

$\boxed{P = 0}$ 1 pt

Final answer should be supported by proper justification/calculation.

Photon 1 through x and 2 through y

$$P = \left| {}_1\langle x| \, {}_2\langle y| \frac{1}{\sqrt{2}}\left(|x\rangle_1 |y\rangle_2 + |y\rangle_1 |x\rangle_2 \right) \right|^2$$ 2 pts

$$= \left| \frac{1}{\sqrt{2}}\left(\underbrace{{}_1\langle x|x\rangle_1}_{=1} \cdot \underbrace{{}_2\langle y|y\rangle_2}_{=1} + \underbrace{{}_1\langle x|y\rangle_1}_{=0} \cdot \underbrace{{}_2\langle y|x\rangle_2}_{=0} \right) \right|^2$$

$P = \left|\frac{1}{\sqrt{2}}\right|^2 \Rightarrow \boxed{P = \frac{1}{2} \text{ or } 50\%}$ 1 pt